ANALOG MONTE CARLO

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1. INTRODUCTION

A simple example clarifies the nature of the Monte Carlo method in its simplest unsophisticated form as a simple analog experiment, which can be carried out in close relationship to the physics involved. Suppose one is asked to determine the value of π. Several approaches can be adopted for this purpose, simpler than the Buffon Needle's approach used by Laplace.

2. A PURE PHYSICS EXPERIMENT

One would get a string of, say, a length:

$$L = 100 \text{ [cm]}.$$

The string would be turned as best achievable into a circle and the diameter of the circle is measured. Suppose that we measure the diameter to the nearest millimeter as:

$$D = 31.8 \text{ [cm]}.$$

To calculate the value of π, we use our knowledge about the relation between the diameter D and the circumference of the circle, which is now equal to the length of the string L, as:

$$\pi = L/D = 100/31.8 = 3.144654088. \qquad (1)$$

This problem can also be solved by a Monte Carlo procedure.

3. A MONTE CARLO ANALOG EXPERIMENT

This is a hypothetical simple experiment based on dart throwing onto a square board of side length 2R with a circle of radius R enclosed in it as shown in Fig. 1. We shoot darts N times at the square randomly without aiming at the circle. We count the number of times n that the darts hit the circle that is enclosed in the square. If the darts throws covered the square uniformly, then from Fig. 1 we can state that:

$$\frac{Number of hit \sin circle}{Tota \ln umber of trials} = Limit_{N \to \infty}(\frac{n}{N}) = \frac{area of circle}{area of square} = \frac{\pi R^2}{4R^2} = \frac{\pi}{4} \qquad (2)$$

From Eqn. 2 we can estimate the value of π as:

$$\pi = 4 Limit_{N \to \infty} \left(\frac{n}{N} \right) \qquad\qquad (3)$$
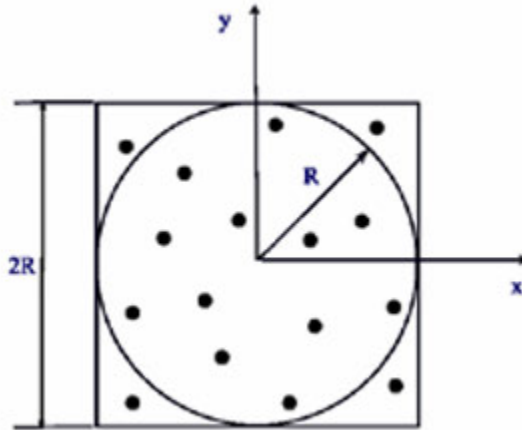
Fig. 1: Analog Monte Carlo dart Experiment.

From Fig. 1, an estimate of π, for a small number of experiments N=16, and n=12 hits, can be estimated as:

$$\pi = 4 \frac{n}{N} = 4 \frac{12}{16} = 3 .$$

It is clear that to get a better estimate of π, we need to carry a large number of experiments N. We thus seek the help of a computer. The issue is how to construct a procedure for a computer simulation of the dart throwing process.

4. A MONTE CARLO COMPUTER EXPERIMENT

Consider the upper right hand quadrant of the circle in Fig.1 as shown in Fig.2. Determine a uniformly distributed random point r_y along the y-axis, and another one r_x along the x-axis. The points r_x and r_y define a point in the plane:

$$(r_x , r_y).$$

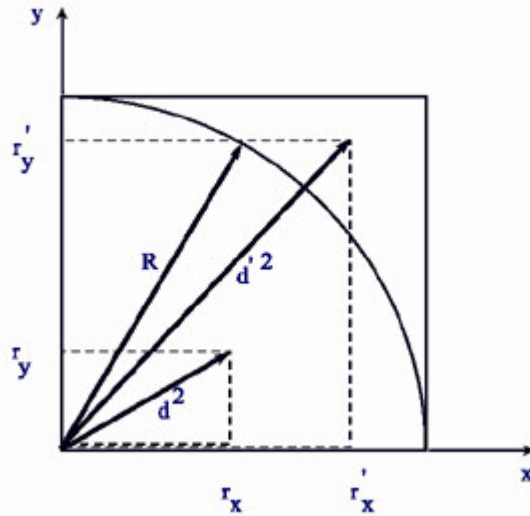Depending on the position of the sampled point,

$$If : d^2 = r_x^2 + r_y^2 \le R^2, \qquad \text{score a hit,}$$

$$If : d'^2 = r_x'^2 + r_y'^2 \ge R^2, \qquad \text{do not score a hit.}$$

Repeating the process N times allows the computation the ratio:

$$S = \frac{Numberofhits}{Numberoftrials} = \frac{n}{N}$$

Fig.2: Quadrant geometry in dart throwing experiment.

From Fig. 2, this quantity is the ratio of the area of the circular quadrant to the area of the fourth of the square and satisfies the relationship:

$$S = \frac{n}{N} = \frac{(\frac{\pi R^2}{4})}{\frac{(2R)^2}{4}} = \frac{\pi}{4} \tag{4}$$

From Eqn. 4 an estimate of π can be obtained as:

$$\pi = 4S = 4\frac{n}{N} \tag{5}$$

5. MONTE CARLO PROCEDURE

The flow chart of the algorithm for the computer simulation of the process of darts throwing is shown in Fig.3, and the listing of the corresponding algorithm is shown in Fig.4.

Two counters for the number of hits and the number of trials are initialized to zero. A point in the square is uniformly sampled. If a test of its position implies a hit, the hit counter is incremented. Another test checks the total n umber of trials. When the maximum number of assigned trials is reached, the value of π is calculated, the results displayed and written on the screen, and the computation is stopped.
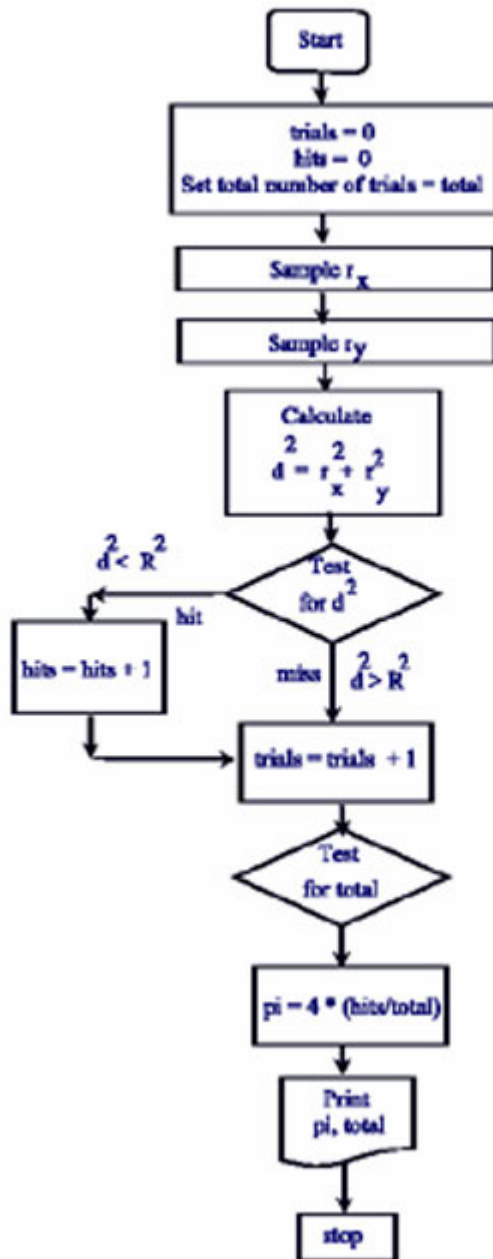
Fig.3: Flow Chart for dart throwing problem.

```
! darts_throwing f90
! Monte Carlo Simulation of darts throwing at a circle enclosed
! in a square to estimate the value of pi
! M. Ragheb, Univ. of Illinois
!
        program darts_throwing
        real(8) x_coordinate,y_coordinate,hits,pi,trials
        real(8) circle_radius,circle_radius_square,position_square
```

```
! Initialization of variables
        trials = 1.0E+09
        ntrials=trials
        square_side_length = 2.0
        circle_radius = square_side_length / 2.0
        circle_radius_square=circle_radius*circle_radius

! Simulation of darts throwing
        do i = 1,ntrials
                call random (rr)
                x_coordinate = circle_radius * rr
                call random (rr)
                y_coordinate = circle_radius * rr
                position_square = ( x_coordinate * x_coordinate + &
        &                          y_coordinate * y_coordinate )

                        if (position_square .LE. circle_radius_square) then
! Score a hit
                        hits = hits + 1.0
                        endif
        end do
! Estimate value of pi
        pi = 4.0 * hits / trials
! Print results
        print *,' Trials      Hits      Pi    '
        print *,'-----------------------------------------'
        print '(3(E15.8,2x))',trials, hits, pi

        end
```

Fig.4: Listing of Monte Carlo procedure for dart throwing problem.

Table 1: Comparison of estimates of the value of π, for different numbers of throws N.

Number of trials, N	Estimate of π Laplace's Buffon Needle Monte Carlo	Estimate of π Darts throwing, Analog Monte Carlo.
10^1	4.999999	2.8000000
10^2	3.703704	3.3200000
10^3	3.194888	3.1480000
10^4	3.193868	3.1344000
10^5	3.147326	3.1474000
10^6	3.138978	3.1442680
10^7	3.140663	3.1425804
10^8	3.141302	3.1416612
10^9	3.141682	3.1416093

6. DISCUSSION

This simple problem shows how a Monte Carlo Procedure involves several distinct basic steps:

1. The definition of an analog numerical experiment equivalent to the physical experiment that we would like to perform.
2. Sampling a distribution function, here a uniform distribution, using random numbers.
3. Estimation of parameters over the performed experiment to determine the quantities of interest.

Estimates of the value of π calculated with the dart-throwing algorithm are shown in Table 1 for different numbers of experiments or trials N. These results are compared to those obtained from the Pascal's Buffon Needle algorithm.

PROBLEMS

1.   Plot the convergence of the value of π, as well as the absolute relative error in the estimate as N becomes larger. Compare the estimates of π as a function of N to those obtained by using the Pascal's Buffon Needle approach. Does the error decrease as the predicted the inverse square root of the number of trials?

2.   Vary the size of the circle and square in the simulation, does that affect the estimates?